SMALL GENERATORS OF COCOMPACT ARITHMETIC
FUCHSIAN GROUPS

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Abstract. In the study of Fuchsian groups, it is a nontrivial problem to
determine a set of generators. Using a dynamical approach we construct for
any cocompact arithmetic Fuchsian group a fundamental region in $\text{SL}_2(\mathbb{R})$
from which we determine a set of small generators.

1. Introduction

Arithmetic Fuchsian groups form a subclass of Fuchsian groups with a special
connection to number theory and the theory of automorphic forms. These groups
are necessarily of finite covolume and thus finitely generated. However, as with gen-
eral Fuchsian groups, it is a nontrivial problem to determine a set of generators.
One way to do this involves constructing a polygonal fundamental domain in $\mathbb{H}^2$
and listing a set of generators corresponding to the side pairings of the polygon.
Johansson, Voight, and Page produced algorithms to determine fundamental do-
mains for groups of units in a maximal order of a quaternion algebra which are
arithmetic Fuchsian (see [6, 17]) or Kleinian (see [15, Chapter 3]). Macasieb deter-
mined fundamental domains for derived arithmetic Fuchsian groups of genus 2 (see
[11]). One could theoretically determine a set of generators for other groups in the
commensurability class, but this seems hard in general.

We also note that Chinburg and Stover [4] obtain bounds for small generators
S-units of division algebras using lattice point methods. However, their generators
are small in the division algebra and their representatives in $\text{GL}_n$ do not necessarily
have small entries.

In this paper, we consider any cocompact arithmetic Fuchsian group and use
dynamical techniques to construct a fundamental region in $\text{SL}_2(\mathbb{R})$ from which we
determine a set of small generators. Our methods are reminiscent of methods used
by Burger and Schroeder (see [3]) and in Page’s thesis (see [15, Chapter 2]).

Although each of our generators will be small, the number of generators could
be quite large. Given a presentation, Voight gives an algorithm (which works for
general finite covolume Fuchsian groups) to construct a new generating set with a
minimal number of generators (see [17]).

1.1. Statement of Results. Let $G = \text{SL}_2(\mathbb{R})$ with a fixed Haar measure $\mu$, and
$\Gamma$ be a cocompact arithmetic Fuchsian group (see subsection 3.1). Our main result
gives a bound on norms of the generators in terms of the degree of the invariant

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trace field $k$, the volume of $G/\Gamma$, and the spectral gap of the Laplace-Beltrami operator on $\mathbb{H}^2/\Gamma$.

**Theorem 1.1.** There exists a constant $C > 0$ depending only on the Haar measure $\mu$ and satisfying the following property. Let $\Gamma$ be a cocompact arithmetic Fuchsian subgroup of $G$ with invariant trace field $k$. Then $\Gamma$ is generated by the finite subset

$$\left\{ \gamma \in \Gamma : \|\gamma\| < C \cdot \left( \mid k : \mathbb{Q} \mid^{-\frac{60}{1 - \sqrt{1 - 4 \lambda_1}} \cdot \text{vol}_\mu(G/\Gamma)^{\frac{6}{1 - \sqrt{1 - 4 \lambda_1}}} \right) \right\}.$$  

Here $\text{vol}_\mu(G/\Gamma)$ is the covolume of $\Gamma$ with respect to the measure $\mu$, and $\lambda = \min\{\frac{1}{4}, \lambda_1(\Gamma)\}$ where $\lambda_1(\Gamma)$ is the smallest non-zero eigenvalue of the Laplace-Beltrami operator on $\mathbb{H}^2/\Gamma$.

Note that over a finite-dimensional space the choice of norm $\|\cdot\|$ does not matter. However, for completeness we take the $L^\infty$-norm. The coefficient $C$ in Theorem 1.1 can be explicitly calculated by following the methods of the proofs. Also, the result can be strengthened for certain families of arithmetic Fuchsian groups $\Gamma$, as we now describe.

**Corollary 1.2.** If $\Gamma$ is a cocompact congruence arithmetic Fuchsian group then it is generated by the finite subset

$$\left\{ \gamma \in \Gamma : \|\gamma\| < C \cdot \left( \mid k : \mathbb{Q} \mid^{-\frac{384}{5}} \cdot \text{vol}_\mu(G/\Gamma)^{\frac{192}{25}} \right) \right\}.$$  

**Corollary 1.3.** If $\Gamma$ is torsion free cocompact arithmetic Fuchsian group then it is generated by the finite subset

$$\left\{ \gamma \in \Gamma : \|\gamma\| < C \cdot \left( \log(\mid k : \mathbb{Q} \mid^{-\frac{180}{1 - \sqrt{1 - 4 \lambda_1}}}) \cdot \text{vol}_\mu(G/\Gamma)^{\frac{6}{1 - \sqrt{1 - 4 \lambda_1}}} \right) \right\}.$$  

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3. Arithmetic Fuchsian groups

We recall the definition and some properties of arithmetic Fuchsian groups.

**3.1. Definition and properties.** Let $k$ be a totally real number field and $A$ a quaternion algebra over $k$ ramified at all archimedean places except one. Let $R_k$ be the ring of integers in $k$, $O$ an $R_k$-order in $A$, and $O^1$ the elements of $A$-norm 1 in $O$. Let $\rho$ be a $k$-embedding of $A$ in $M_2(\mathbb{R})$, $2 \times 2$ matrices over $\mathbb{R}$. Then $\rho(O^1)$ is a Fuchsian group of finite covolume. By definition, a subgroup $\Gamma$ in $\text{SL}_2(\mathbb{R})$ is an arithmetic Fuchsian group if it is commensurable, up to conjugation, with some such $\rho(O^1)$.

An arithmetic Fuchsian group $\Gamma$ is called a congruence subgroup if there is a maximal order $O$ and an integral 2-sided $O$-ideal $I$ of $O$ in $A$ such that $\Gamma$ contains $\rho(O^1(I))$ where $O^1(I) = \{a \in O^1 : a - 1 \in I\}$.

The field $k$ in the definition of an arithmetic Fuchsian group $\Gamma$ is a commensurability invariant and is recovered from $\Gamma$ by $k\Gamma = \mathbb{Q}(\text{tr}(\Gamma(2)))$, where $\Gamma(2)$ is
the subgroup generated by the squares of elements in $\Gamma$. The number field $k\Gamma$ is the invariant trace field of $\Gamma$. The quaternion algebra is also recovered from $\Gamma$ by $A\Gamma = \{ \sum a_i \gamma_i : a_i \in k\Gamma, \gamma_i \in \Gamma(2) \}$. For the remainder of the paper, we assume $\Gamma$ is cocompact. The group $\Gamma$ is cocompact when either $k \neq \mathbb{Q}$ or when $k = \mathbb{Q}$ and $A$ is a division algebra. We note also that for any $\gamma \in \Gamma$, $\text{tr}(\gamma)$ is an algebraic integer of degree at most 2 over $k\Gamma$.

For more on arithmetic Fuchsian groups, including proofs of the statements mentioned, the reader is referred to [12, Chapter 8].

3.2. Lehmer’s and Salem’s conjectures. The Mahler measure of an algebraic integer $\theta$ is defined by

\begin{equation}
M(\theta) = |a_0| \prod \max(1, |\theta_i|)
\end{equation}

where $a_0$ is the leading coefficient of the minimal polynomial for $\theta$ over $\mathbb{Z}$ and $\theta_i$ ranges over the algebraic conjugates of $\theta$.

In 1933 Lehmer posed the following conjecture:

**Conjecture 3.1.** [10] There exists a universal lower bound $m > 1$ such that $M(\theta) \geq m$ for all $\theta$ which are not roots of unity.

A Salem number is a real algebraic integer $\theta > 1$ such that $\theta^{-1}$ is a conjugate and all other conjugates lie on the unit circle. For Salem numbers, Lehmer’s conjecture restricts to the Salem conjecture.

**Conjecture 3.2.** There exists $m_S > 1$ such that if $\theta$ is a Salem number then $M(\theta) \geq m_S$.

The Salem conjecture is equivalent to the short geodesic conjecture of arithmetic 2-manifolds, which states that there is a universal lower bound for translation length for hyperbolic elements in arithmetic Fuchsian groups (see [14]). Assuming the Salem conjecture, the bounds in Theorem 1.1 can be modified to depend only on the covolume and the spectral gap whenever the group $\Gamma$ contains no elliptic elements.

**Corollary 3.3.** If the Salem conjecture is true and $\Gamma$ is a torsion free, cocompact, arithmetic Fuchsian subgroup of $G$. Then $\Gamma$ is generated by the finite subset

\begin{equation}
\left\{ \gamma \in \Gamma : \|\gamma\| < C \cdot (\log(m_S))^{1/\sqrt{\log(2d)}} \cdot \text{vol}(G/\Gamma)^{1/\sqrt{\log(2d)}} \right\}.
\end{equation}

4. Preparation

4.1. Injectivity radius. Throughout this section, let $d$ be the degree of the the invariant trace field of $\Gamma$.

**Lemma 4.1.** Let $\Gamma$ be a cocompact arithmetic Fuchsian group with invariant trace field of degree $d$ and let

\begin{equation}
W_d = \left\{ g \in G : 2 \cos\left(\frac{\pi}{2d}\right) < |\text{tr}(g)| < 2 \cosh\left(\frac{1}{16} \left(\log(\log(2d))^{3/2} \cdot \log(2d)\right)^{3/2}\right) \right\}.
\end{equation}

Then for any $g \in G$, $g\Gamma g^{-1} \cap W_d = \{1\}$
Proof. By the cocompactness condition, we consider only hyperbolic and elliptic elements in $\Gamma$.

If $\gamma$ is hyperbolic then also $\gamma^2$ is hyperbolic, with $|\text{tr}(\gamma^2)| > 2$ and $\text{tr}(\gamma^2) \in k\Gamma$. Let $u_\gamma$ be the root of the characteristic polynomial $x^2 - \text{tr}(\gamma)x + 1$ of $\gamma$ which lies outside the unit circle (respectively, let $u_{\gamma^2}$ be the root of the characteristic polynomial $x^2 - \text{tr}(\gamma^2)x + 1$ of $\gamma^2$ outside the unit circle). We have $u_{\gamma^2}$ real and exactly two of its algebraic conjugates lie off the unit circle, namely $u_{\gamma^2}$ and $\overline{u_{\gamma^2}}$ (see e.g. [12, Lemma 12.3.1]). Therefore, $\log M(u_{\gamma^2}) = \log |u_{\gamma^2}| = 2\log |u_\gamma|$ where $M(u_{\gamma^2})$ is the Mahler measure (see Equation 3.1). As a hyperbolic element, $\gamma$ acts on the hyperbolic plane by translating along its invariant axis a distance of $2\log |u_\gamma|$. It then follows that $|\text{tr}(\gamma)| = 2\cosh \left( \frac{1}{2} \log M(u_{\gamma^2}) \right)$ (see e.g. [12, Lemma 12.1.2]). Since $u_{\gamma^2}$ lies in an extension of degree 2 over $k\Gamma$, it has algebraic degree bounded above by $2d$. We now apply the main theorem of [18] which states that any algebraic integer $\theta$ of degree at most $2d$ has

$$\log M(\theta) > \frac{1}{4} \left( \frac{\log(\log(2d))}{\log(2d)} \right)^3$$

(4.2)

to get the following bound for for the trace of a hyperbolic $\gamma \in \Gamma$

$$|\text{tr}(\gamma)| > 2\cosh \left( \frac{1}{16} \left( \frac{\log(\log(2d))}{\log(2d)} \right)^3 \right).$$

(4.3)

If $\gamma$ is elliptic then by the discreteness of $\Gamma$, $\gamma$ has order $m$ and thus eigenvalues $\omega$ and $\omega^{-1}$ with $|\omega + \omega^{-1}| = |\text{tr}(\gamma)| < 2$ and such that $\omega$ is a primitive $2m$-th root of unity. Since $\text{tr}(\gamma)$ lies in a field of degree at most 2 over $k\Gamma$, $(\omega + \omega^{-1})$ has algebraic degree of at most $2d$, or equivalently, $\omega$ has algebraic degree of at most 4d. Then $|\text{tr}(\gamma)| = |u_\gamma + u_\gamma^{-1}|$ is bounded above by $|e^{\frac{2\pi i}{2d}} + e^{-\frac{2\pi i}{2d}}| = 2\cos(\frac{\pi}{2d})$.

Take $W_d$ as in Equation 4.1. Since the trace is invariant under conjugation, for any $g \in G$ we have $g\Gamma g^{-1} \cap W_d = \{1\}$.

The injectivity radius of $G/\Gamma$ at a point $x = h\Gamma \in G/\Gamma$ is related to the distance between the identity element and the nearest point in the lattice $h\Gamma h^{-1}$. Since the trace of a matrix is invariant under conjugation, the bounds in Lemma 4.1 for the traces of elements in any conjugate of $\Gamma$ determine an injectivity radius for $G/\Gamma$.

For any $\eta > 0$ denote the ball of radius $\eta$ centered at the identity $1 \in G$ by

$$B_G(\eta) = \{g \in G : \|g - 1\| < \eta \}.$$

Lemma 4.2. There exists $\delta = \delta(d) = cd^{-2} > 0$ with $c$ depending only on the Haar measure $\mu$, such that the map $B_G(\delta) \to G/\Gamma$ given by $g \mapsto g.x$ is injective for any $x \in G/\Gamma$.

Proof. Let

$$\delta_0 = \min \left\{ \cosh \left( \frac{1}{16} \left( \frac{\log(\log(2d))}{\log(2d)} \right)^3 \right) - 1, 1 - \cos \left( \frac{\pi}{2d} \right) \right\}.$$ 

(4.4)

Then for any $0 < \delta_1 < \delta_0$ and $g \in B_G(\delta_1)$,

$$|g_{1,1} - 1| < \delta_1 \quad \text{and} \quad |g_{2,2} - 1| < \delta_1,$$

where $g_{1,1}$ and $g_{2,2}$ are the $(1, 1)$ and $(2, 2)$ matrix entries of $g$. Then $|\text{tr}(g) - 2| < 2\delta_1$ and $g \in W_d$. 


By the Taylor series expansion of $1 - \cos(\pi/2d)$,
\begin{equation}
1 - \cos\left(\frac{\pi}{2d}\right) \approx \frac{\pi^2}{8d^2} - \frac{\pi^4}{384d^4}.
\end{equation}
Since $1 - \cos\left(\frac{\pi}{2d}\right)$ goes to 0 at a faster rate than $\cosh\left(\frac{1}{16}\left(\frac{\log(\log(2d))}{\log(2d)}\right)^3\right) - 1$ goes to 0, there exists a positive constant $c$ depending only on the measure $\mu$ such that $\delta(d) := c \cdot d^{-2} < \delta_0$ and such that for all $g_1, g_2 \in B_G(\delta)$, we have $g_1^{-1}g_2 \in B_G(\delta_0)$.

For any $x = h\Gamma \in G/\Gamma$, if $g_1, g_2 \in B_G(\delta)$ and $g_1x = g_2x$ then $g_1^{-1}g_2 \in h\Gamma h^{-1}$. Hence $g_1^{-1}g_2 \in W_d$ and by Lemma 4.1 it must be that $g_1 = g_2$. \hfill \square

Remark 4.3. When $\Gamma$ is torsion free, only the traces of hyperbolic elements are relevant. In this case the constant $\delta(d)$ in Lemma 4.2 can be improved to $c\left(\frac{\log(\log(2d))}{\log(2d)}\right)^6$.

For a cleaner statement, we use the estimate $\log(d)^{-1} \ll \left(\frac{\log(\log(2d))}{\log(2d)}\right)^6$.

If the Salem conjecture is true, there is a universal lower bound for the trace of a hyperbolic element in an arithmetic Fuchsian group. Then for $m_S$ the lower bound satisfying the Salem conjecture, the constant $\delta$ can be improved to $c\log^2(m_S)$ for all torsion free $\Gamma$. In particular, this constant would be independent of $d$.

4.2. Spectral gap. Let $\Gamma$ be a cocompact Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ from the right. Let $\lambda_1(\Gamma)$ be the first non-zero eigenvalue of the Laplacian operator on the locally symmetric space $\mathbb{H}^2/\Gamma$. Denote by $\lambda = \lambda(\Gamma) := \min\{\frac{1}{4}, \lambda_1(\Gamma)\}$. Let
\begin{equation}
\alpha_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{sl}_2(\mathbb{R}).
\end{equation}

Denote by $\mu_{G/\Gamma}$ the $G$-invariant probability measure supported on $G/\Gamma$ for which, whenever $X \subset G/\Gamma$ is a measurable set, we have $\text{vol}_\mu(X) = \text{vol}_\mu(G/\Gamma) \cdot \mu_{G/\Gamma}(X)$.

Lemma 4.4. Let $\Gamma$ be a cocompact lattice in $G$. Then for any smooth functions $\varphi, \psi$ in $L^2_h(G/\Gamma) = \{ f \in L^2(\Gamma) : f d\mu_{G/\Gamma} = 0 \}$ it holds
\begin{equation}
\| \langle \alpha_t, \varphi \rangle, \psi \rangle \| \ll (e^{-t/3})^{1/2} \cdot \|D\varphi\| \cdot \|D\psi\|.
\end{equation}

Proof. This lemma follows from work of Ratner in [16] but is given explicitly in [13 Corollary 2.1] and with explicit constant. We use the estimate $t(e^{-t/2}) \ll (e^{-t/3})$ for a cleaner statement. \hfill \square

4.3. Translates. Given any two points in the $\Gamma$, some element $h \in G$ will translate one to the other. We use the injectivity radius, the spectral gap, and the covolume to give an upper bound on the norm of some such element $h$.

Theorem 4.5. For any $g_1\Gamma, g_2\Gamma \in G/\Gamma$ there is some $h \in G$ and positive number $c$ such that $h.g_1\Gamma = g_2\Gamma$ and
\begin{equation}
\|h\| < c \cdot \text{vol}_\mu(G/\Gamma)^{\frac{3}{16\cdot 4\pi}} \cdot \delta^{\frac{15}{16\cdot 4\pi}},
\end{equation}
where $\delta$ is in Lemma 4.3 and $\lambda$ is as in subsection 4.2.

Proof. Let $\varphi$ and $\psi$ be nonnegative, smooth bump functions supported in $B_G(\delta).g_1\Gamma$ and $B_G(\delta).g_2\Gamma$ respectively, and satisfying
\begin{equation}
\mu_{G/\Gamma}(\varphi) = \mu_{G/\Gamma}(\psi) = \text{vol}_\mu(G/\Gamma)^{-1}, \quad \|D\varphi\|, \|D\psi\| \ll \text{vol}_\mu(G/\Gamma)^{-\frac{1}{2}} \cdot \delta^{-\frac{1}{2}}.
\end{equation}
These functions can be constructed as in [9 Lemma 2.4.7]. Applying Lemma 4.4 to the functions \( \varphi - \mu_{G/\Gamma}(\varphi) \) and \( \psi - \mu_{G/\Gamma}(\psi) \), we get
\[
\langle \alpha_t, \varphi, \psi \rangle - \mu_{G/\Gamma}(\varphi) \cdot \mu_{G/\Gamma}(\psi) \ll (e^{-t/5})^{1-i\sqrt{1+3\alpha}} \cdot ||D_1\varphi|| \cdot ||D_1\psi||.
\]
Then for \( t > 0 \) such that
\[
\|\alpha_t\| \leq e^t \ll \text{vol}_\mu(G/\Gamma) \frac{1-3}{1-i\sqrt{1+3\alpha}} \cdot \delta^{1-i\sqrt{1+3\alpha}},
\]
we must have \( \langle \alpha_t, \varphi, \psi \rangle \neq 0 \), since \( \mu_{G/\Gamma}(\varphi) \cdot \mu_{G/\Gamma}(\psi) \) will dominate the error term in Equation 4.10. Therefore, the set
\[
(\alpha_t, \supp(\varphi) \cap \supp(\psi)) \subset (\alpha_t, (B_G(\delta), g_1 \Gamma) \cap (B_G(\delta), g_2 \Gamma))
\]
is not empty. There is some \( h \in B_G(\delta)^{-1} \cdot \alpha_t \cdot B_G(\delta) \) for which \( h \cdot g_1 \Gamma = g_2 \Gamma \). \( \square \)

5. PROOF OF RESULTS AND CONCLUDING REMARKS

Proof of Theorem 1.1 Define the set
\[
U = \left\{ h \in G : \|h\| \leq \text{vol}_\mu(G/\Gamma) \frac{1-3}{1-i\sqrt{1+3\alpha}} \cdot \delta^{1-i\sqrt{1+3\alpha}} \right\}.
\]
By Theorem 4.5, for any \( g \in G \), there is some \( h \in U \) such that \( h \cdot g \cdot g_2 \Gamma = g_1 \Gamma \). That is, we can write \( g \) as \( h \cdot g \) for some \( g_2 \cdot g_1 = \gamma \in \Gamma \). Therefore, we have constructed an open set \( U \subset G \) such that \( U \cdot g \cdot g_2 \Gamma = g_1 \Gamma \). It follows that \( \{ \gamma \in \Gamma : U \cdot \gamma \cap U \neq \emptyset \} \) generates \( \Gamma \) (see e.g. [2, Lemma 6.6]). Then \( \Gamma \) is generated by the bigger set \( U^{-1}U \cdot g_2 \Gamma \), whose elements have norm bounded above by \( \text{vol}_\mu(G/\Gamma) \frac{1-3}{1-i\sqrt{1+3\alpha}} \cdot \delta^{1-i\sqrt{1+3\alpha}} \).

The proof of Theorem 1.1 then follows by replacing \( \delta \) with \( c[k : \mathbb{Q}]^{-2} \), as given in Lemma 4.2 (where \( d = [k : \mathbb{Q}] \)). \( \square \)

Proof of Corollary 1.2 The proof of Corollary 1.2 follows from the well-known result that
\[
\lambda_1(\Gamma) \geq 975/4096
\]
for every congruence arithmetic Fuchsian group. For the sake of reader’s convenience, we shall briefly explain Equation 5.2 in the case when \( k \Gamma = \mathbb{Q} \).

By the Jacquet-Langlands correspondence (see [14 Chapter 16]), if \( \Gamma \) is a congruence arithmetic Fuchsian group with \( k \Gamma = \mathbb{Q} \) then \( \lambda_1(\Gamma) \geq \inf \{ \lambda_1(\Lambda(N)) : N \in \mathbb{N} \} \), where \( \Lambda(N) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv 1 \mod N \} \). As of this writing we have the estimate, which is due to Kim-Sarnark [8],
\[
\inf \{ \lambda_1(\Lambda(N)) : N \in \mathbb{N} \} \geq 975/4096.
\]
We note that, for the case when \( k \Gamma \) is a number field, Equation 5.2 follows from the Jacquet-Langlands correspondence together with the bounds on the Ramanujan conjecture over number fields (see [1]). \( \square \)

Proof of Corollary 1.3 and Corollary 3.3 The proof of Corollary 1.3 follows from the proof of Theorem 1.1 by replacing \( \delta \) with \( c \log(d)^{-4} \) as in Remark 4.3. Furthermore, if we assume the Salem conjecture is true, then Corollary 3.3 will follow by setting \( \delta = c \log^2(m_S) \) as in Remark 4.3. \( \square \)

Remark 5.1 The arguments in the proof of Theorem 1.1 can be modified to give a similar result for cocompact arithmetic Kleinian groups. All lemmas will go through except that one would need an analogue of Lemma 4.4. An analogue of Lemma 4.4 (or [13, Corollary 2.1]) does hold. However, more work would need to be done to
get an explicit rate of the decay of correlations for functions on $\text{SL}_2(\mathbb{C})/\Gamma$, from the $\lambda_1$ of the corresponding hyperbolic 3-manifold $\text{SU}(2)\backslash \text{SL}_2(\mathbb{C})/\Gamma$. At this time, we are not aware of such a statement in the current literature.

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